

Acta Mechanica, Vol. 150 (2001), pp.23-37

**Fully Developed Flow of a Modified Second Grade Fluid with
Temperature Dependent Viscosity**

by

Mehrdad Massoudi and Tran. X. Phuoc
U.S. Department of Energy
National Energy Technology Laboratory
P. O. Box 10940
Pittsburgh PA. 15236
USA

Massoudi@netl.doe.gov

Abstract

In this paper we will study the fully developed flow of a modified, and sometimes referred to as the generalized, second grade fluid down an inclined plane. The reasons for using such a model for flow of non-Newtonian fluids are (i) the capability of predicting the normal stress differences and (ii) allowing for the possibility of shear dependent viscosity. The boundary value problem is solved numerically, and the special case of constant viscosity amends itself an exact solution (as previously reported in literature) which serves as a test case to check the accuracy of our numerical scheme. The velocity and temperature profiles are obtained for various dimensionless numbers, for the case where the viscosity is also a function of temperature.

Keywords: Non-Newtonian fluids, film flow, temperature dependent viscosity

Introduction

Two distinct features of many non-Newtonian fluids are either the fact that many of them exhibit normal stress differences or the fact that their viscosity depends on shear rate. Of course, there are other features such as yield stress, time-dependency, history effects and other non-linear issues; in these cases more complex constitutive relations should be used. Perhaps the simplest model which can predict the normal stress differences is the second grade fluid, or the Rivlin-Erickson fluid of grade two [cf. Rivlin and Ericksen (1955), or Truesdell and Noll (1992)]. This model has been used and studied extensively [cf. Dunn and Fosdick (1974)], and is a special case of fluids of differential type [cf. Dunn and Rajagopal (1995)]. Though this model is relatively simple (even though higher order terms are introduced) its viscosity is assumed to be constant, and thus it cannot be used for fluids where experimental data indicate shear dependent viscosity. At the same time, one of the most widely used non-Newtonian models in the field of engineering is the so-called “Power-law” model [cf. Bird, et al. (1977), or Slattery (1999)], which allows for the viscosity to depend on the velocity gradient. This model has been extensively used in coal-water slurries (cf. Shook and Roco (1991)). However, this model cannot predict the normal stress effects which could lead to phenomena like “die-swell,” and “rod-climbing,” [cf. Schowalter (1979)] which are manifestations of the stresses that develop orthogonal to planes of shear.

One of the main areas of interest in energy related processes, such as power plants, atomization, alternative fuels, etc., is the use of slurries, specifically coal-water or coal-oil slurries, as the primary fuel. Some studies indicate that the viscosity of coal-water mixtures depends not only on the volume fraction of solids, and the mean size, and the size distribution of the coal, but also on

the shear rate, since the slurry behaves as a shear-thinning fluid (cf. Roh, et al. (1995), Papachristodoulou and Trass (1987), Tsai and Knell (1986)). At the same time, there are studies which indicate that preheating the fuel results in better performance (cf. Tsai, et al (1988), Saeki and Usui (1995)), and as a result of such heating, the viscosity changes. A similar situation, i.e, efficient heating or cooling of a liquid occurs in the flow of a thin film along a solid surface which is kept at a constant temperature. Astarita et al (1964) studied the fully developed flow of a non-Newtonian film along a plane surface. Gupta (1967) studied the flow of a second grade fluid down an inclined plane. He also performed a linearized stability analysis. Stability analysis of non-Newtonian fluids, especially second grade fluids has received much attention (cf. Dunn and Fosdick (1974), Straughan (1983), and Straughan (1992)). In recent years, heat transfer to a falling fluid film has been a subject of extensive research (cf. Rao (1999), Andersson and Shang (1998), Shang and Andersson (1999)). In most of these studies, the non-Newtonian fluid is represented using the power-law model. There are very few studies where the effects of viscous dissipation is included (cf. Barletta (1997)), even though this has been shown to be very important in many cases such as polymer processing (cf. Winter (1977)).

Man et al (1987) and Man (1992) proposed models where the shear viscosity (and also the normal stress coefficients) of a second grade fluid would also depend on the shear rate. Therefore this new ‘modified’ or ‘generalized’ second grade fluid is not only capable of predicting normal stress differences, but also it can be used for shear- thinning and shear-thickening fluids. Gupta and Massoudi (1993) proposed a model based on Man’s work (1992) where in addition to the above effects, they suggested that the shear viscosity can also depend on temperature. Therefore, their model becomes very useful for cases when the effect of viscous dissipation and the effect of temperature on viscosity cannot be ignored. Franchi and Straughan (1993) have done a stability analysis for a modified second grade fluid where the viscosity is assumed to be a linear function of temperature.

In this paper we will study the fully developed flow of a non-Newtonian fluid down a heated inclined plane. The constitutive relation is that of the modified second grade fluid where the viscosity is assumed to depend on temperature [cf. Gupta and Massoudi (1993)].

Constitutive Relation

Fluids of differential type form an important class of non-Newtonian fluids. The thermodynamics and stability of fluids of second and third grade fluids have been studied extensively by Dunn and Fosdick (1974), and Fosdick and Rajagopal (1980), and a recent review article by Dunn and Rajagopal (1995) sheds light on many of the interesting and challenging issues in the fluids of differential type. The generalized (or modified) form of the second grade fluid is where the shear viscosity (or all the rheological properties, i.e., the normal stress coefficient, as well) depend on the shear rate.

The constitutive relation for the modified second grade fluid is given by [cf. Man (1992)]

$$\mathbf{T} = -p\mathbf{1} + \mu\pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \quad (1)$$

where \mathbf{T} is the Cauchy stress tensor; p is the indeterminate part of the stress due to the constraint of incompressibility, μ is the coefficient viscosity, and α_1 and α_2 are material moduli usually referred to as the normal stress coefficients. The kinematical tensors \mathbf{A}_1 and \mathbf{A}_2 are the first and the second Rivlin-Ericksen tensors (cf. Rivlin and Ericksen (1955)), respectively, and are given by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T \quad (2)$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1 \quad (3)$$

$$\mathbf{L} = \nabla\mathbf{v} \quad (4)$$

where \mathbf{v} denotes the velocity field, ∇ is the gradient operator, and d/dt is the material time derivative which is defined as

$$\frac{d(.)}{dt} = \frac{\partial(.)}{\partial t} + [\nabla(.)]\mathbf{v} \quad (5)$$

where $\partial/\partial t$ is the partial time derivative. In Eqn.(1), μ , m , α_1 and α_2 are material functions which only depend on temperature; when $m < 0$, the fluid is “shear-thinning,” while if $m > 0$, the fluid is “shear-thickening.” When $m = 0$, Eqn (1) reduces to the standard second grade fluid of Rivlin-Ericksen. At the same time, if $\alpha_1 = \alpha_2 = 0$, Eqn (1), reduces to the generalized power-law model. In Eqn (1), $\pi \equiv (\frac{1}{2}) \text{tr} (\mathbf{A}_1^2)$. The constitutive equation (1), proposed by Man, et al (1985) has been used by Man (1992) to study the nonsteady channel flow of ice and by Gupta and Massoudi (1993) to study the flow between two heated plates.

Governing Equations

One of the most widely studied problems, due to its significant industrial applications and ease of computation, is the flow down an inclined plane. When the fluid is assumed to be represented by a power-law model, with a constant viscosity, and where fully developed conditions are assumed, it is possible to obtain an exact solution for the velocity field (cf. Bird, et al. (1977, p. 217)). For the general two dimensional cases, including heat transfer effects, Andersson and co-workers have studied this problem extensively. The stability of a liquid film, where the constitutive equation is that of a second grade fluid was studied by Gupta (1967), where again for a fully developed condition an exact solution was obtained. Recently Akyildiz (1998) has studied the flow of a third grade fluid flowing down a vertical wall which is oscillating longitudinally. In most of the studies, the effects of temperature dependent viscosity and viscous dissipation are ignored. In the present study, we will try to look into these issues, assuming that the constitutive relation for the stress is that of a modified second grade fluid, with a temperature dependent viscosity.

For a fully developed flow, we seek velocity and temperature fields of the form

$$\begin{aligned} \mathbf{v} &= u(y)\mathbf{i} \\ \theta &= \theta(y) \end{aligned} \tag{6}$$

Where \mathbf{i} is the unit vector in the x-direction (the direction of the flow), while y is normal to the inclined plane, with α being the angle of inclination. Substituting Eqns (1) and (6) into the balance of linear momentum

$$\rho \frac{d\mathbf{v}}{dt} = \text{div } \mathbf{T} + \rho \mathbf{b} \tag{7}$$

and using the fact that the fluid can undergo only isochoric motion (incompressibility constraint), i.e. $\text{div } \mathbf{v} = 0$, we obtain

$$\frac{\partial}{\partial x} \left[-p + \alpha_2 \left(\frac{du}{dy} \right)^2 \right] + \frac{\partial}{\partial y} \left(\mu \left| \frac{du}{dy} \right|^2 \right)^{m/2} \frac{du}{dy} + \rho g \sin \alpha = 0 \tag{8}$$

$$\frac{\partial}{\partial y} \left[-p + (2\alpha_1 + \alpha_2) \left(\frac{du}{dy} \right)^2 \right] - \rho g \cos \alpha = 0 \quad (9)$$

$$\frac{\partial p}{\partial z} = 0 \quad (10)$$

If we define a modified pressure p^* through [cf. Rajagopal and Gupta (1984)]

$$p^* = p - (2\alpha_1 + \alpha_2) \left(\frac{du}{dy} \right)^2 \quad (11)$$

Then Eqns (8)-(10) become

$$\frac{\partial p^*}{\partial x} = \frac{\partial}{\partial y} \left[\mu \left(\left| \frac{du}{dy} \right|^2 \right)^{m/2} \frac{du}{dy} \right] + \rho g \sin \alpha \quad (12)$$

$$\frac{\partial p^*}{\partial y} = -\rho g \cos \alpha \quad (13)$$

$$\frac{\partial p^*}{\partial z} = 0 \quad (14)$$

If in Eqns (12)-(13), we set $m = 0$, we obtain the equations of motion for a second grade fluid down an inclined plane given by Gupta (1967). For a pressure-driven flow, using Eqn (11) would indicate that $\partial p^*/\partial y = \partial p^*/\partial z = 0$ and thus Eqn (12) can be integrated directly. In the present problem, however, the pressure gradient in the normal direction is not equal to zero. Let us look at the boundary conditions for this problem. At the surface of the plane, i.e., at $y = 0$, we impose the no-slip condition. That is

$$\text{at } y = 0; \quad u=0 \quad (15)$$

At the free surface, we impose the traction-free condition. That is

$$\text{at } y = h; \quad t_x = t_y = 0 \quad (16)$$

Now,

$$\mathbf{t} = \mathbf{T}^T \mathbf{n} \quad (17)$$

which implies

$$(t_x)_{y=h} = (T_{xy}n_y)_{y=h} = \mu \left[\left| \frac{du}{dy} \right|^2 \right]^{m/2} \frac{du}{dy} = 0 \quad (18)$$

and

$$(t_y)_{y=h} = (T_{yy}n_y)_{y=h} = -p + (2\alpha_1 + \alpha_2) \left(\frac{du}{dy} \right)^2 = 0 \quad (19)$$

Comparing (19) and (11) indicates that

$$p^* = 0 \quad \text{at the free surface} \quad (20)$$

Now, Eq. (13) implies that for a given angle of inclination, α , we have

$$\frac{\partial p^*}{\partial y} = K_1 \quad (21)$$

where K_1 is a constant. Integrating Eqn. (21), we have

$$p^* = K_1 y + C_1(x) \quad (22)$$

Applying Eqn. (20) to Eqn. (22), we have

$$(p^*)_{y=h} = 0 = K_1 h + C_1(x) \rightarrow C_1(x) = -K_1 h = \text{const} \quad (23)$$

Now, differentiating Eq. (22) with respect to x we have

$$\frac{\partial p^*}{\partial x} = 0 \quad (24)$$

Therefore Eq. (12) becomes

$$\frac{\partial}{\partial y} \left(\mu \left[\left| \frac{du}{dy} \right|^2 \right]^{m/2} \frac{du}{dy} \right) = -\rho g \sin \alpha \quad (25)$$

Integrating with respect to y, we have

$$\mu \left(\left| \frac{du}{dy} \right|^2 \right)^{m/2} \frac{du}{dy} = -\rho g y \sin \alpha + C_2 \quad (26)$$

Now, applying Eqn. (18), we have

$$(t_x)_{y=h} = 0 \rightarrow -\rho g h \sin \alpha + C_2 = 0 \rightarrow C_2 = +\rho g h \sin \alpha \quad (27)$$

and therefore, Eqn (26) becomes:

$$\mu \left| \frac{du}{dy} \right|_m \frac{du}{dy} = -\rho g (y-h) \sin \alpha \quad (28)$$

In order to solve this equation, we need to know how the viscosity changes as a function of temperature, and therefore we also have to consider the balance of energy which is

$$\rho \frac{d\epsilon}{dt} = T L - \text{div } \mathbf{q} + \rho r \quad (29)$$

where ϵ is the specific internal energy, \mathbf{q} is the heat flux vector, and r is the radiant heating. We assume that \mathbf{q} is given by the Fourier's conduction law

$$\mathbf{q} = -k \nabla \theta \quad (30)$$

where θ is the temperature and k is thermal conductivity, which is assumed to be constant. Now the specific internal energy ϵ is related to the specific Helmholtz free energy through [(cf. Dunn and Fosdick (1974))]:

$$\epsilon = \psi + \theta \eta = \epsilon(\theta, A_1, A_2) = \hat{\epsilon}(y) \quad (31)$$

where η is the specific entropy. It follows from Eqn. (6) that

$$\frac{d\epsilon}{dt} = 0 \quad (32)$$

and the energy equation reduces to [cf. Gupta and Massoudi (1993)]

$$\mu(\theta) \left| \frac{du}{dy} \right|^{m+2} + k \frac{d^2\theta}{dy^2} = 0 \quad (33)$$

The boundary conditions for this equation are:

$$\theta = \theta_1 \quad \text{at } y = 0 \quad (34)$$

$$\theta = \theta_2 \quad \text{at } y = h \quad (35)$$

where $\theta_2 > \theta_1$. The first term in Eqn (33) represents viscous dissipation which could be very important for the flow of many polymers and viscous oils. The basic equations for the present problem are the momentum equation, given by Eqn (28) subject to the boundary condition (15), and the energy equation, given by Eqn (33) subject to the boundary conditions (34) and (35). These equations are coupled and the form of the momentum equation depends on the form of the viscosity function. We will discuss these and the dimensionless numbers pertinent to this problem in the next section.

Nondimensionalization

Let us introduce the following non-dimensional parameters

$$\begin{aligned} \theta^* &= \frac{\theta - \theta_1}{\theta_2 - \theta_1}; & \mu^* &= \frac{\mu}{\mu_o} \\ \xi &= \frac{y}{h}; & u^* &= \frac{u}{V} \end{aligned} \quad (36)$$

Where μ_o and V are some reference viscosity and velocity respectively. With these, the dimensionless form of the energy equation becomes

$$\frac{d^2\theta^*}{d\xi^2} + \mu^* \Gamma \left| \frac{du^*}{d\xi} \right|^{m+2} = 0 \quad (37)$$

Where

$$\Gamma = \mu_o \left| \frac{V}{h} \right|^{m+2} \frac{h^2}{k(\theta_2 - \theta_1)} \quad (38)$$

and the boundary conditions become

$$\theta(0) = 0; \quad \theta(1) = 1 \quad (39)$$

Now, Γ can be re-written as

$$\Gamma = \frac{V^2}{c_p(\theta_2 - \theta_1)} \frac{c_p \mu}{k} \left(\frac{V}{h} \right)^m \quad (40)$$

where we can define Eckert and Prandtl numbers (cf. Etemad and Mujumdar (1995)):

$$Ec = \frac{V^2}{c_p(\theta_2 - \theta_1)} \quad (41)$$

$$Pr = \frac{c_p \mu}{k} \left(\frac{V}{h} \right)^m \quad (42)$$

and $\Gamma = EcPr$ is also known as the Brinkman number.

For the viscosity, we assume that the fluid obeys the Reynolds viscosity law (used in many lubrication problems (cf. Reynolds (1886), Szeri and Rajagopal (1985))), where

$$\mu^* = e^{-M\theta^*}; \quad M = n(\theta_2 - \theta_1) \quad (43)$$

where n is a constant. With this, the dimensionless form of the momentum equation (Eqn. 28) becomes

$$\frac{du^*}{d\xi} = \beta^{\frac{1}{1+m}} \left[(1-\xi)e^{M\theta^*} \right]^{\frac{1}{1+m}} (\sin\alpha)^{\frac{1}{1+m}} \quad (44)$$

and Eqn. (37) now becomes

$$\frac{d^2\theta^*}{d\xi^2} + \Gamma\beta^{\frac{m+2}{m+1}} e^{\frac{M\theta^*}{1+m}} (1-\xi)^{\frac{m+2}{m+1}} = 0 \quad (45)$$

where

$$\beta = \frac{\rho g h^{m+2}}{\mu_o V^{m+1}} \quad (46)$$

$$\beta' = \frac{\rho g h^{m+2}}{\mu_o V^{m+1}} \sin\alpha \quad (47)$$

Now, the boundary condition for velocity in Eqn (44) becomes

$$u^*(0) = 0 \quad (48)$$

recalling that the condition at the free surface has already been used. Now β can be re-written as:

$$\beta = \left(\frac{gh}{V^2} \right) \frac{\rho h^{m+1} V^{1-m}}{\mu_o} \quad (49)$$

where we can define

$$Fr = \frac{V^2}{gh} \quad (50)$$

$$Re = \frac{\rho h^{m+1} V^{1-m}}{\mu_o} \quad (51)$$

And therefore, β is the ratio of the viscous effects to the gravitational effects. For power-law fluids, the Reynolds number is typically given as:

$$Re = \frac{\rho L^n V^{2-n}}{K} \quad (52)$$

where n is the power-law index, and K is the coefficient of consistency (cf. Andersson and Shang (1998)). Comparing Eqn. (52) with Eqn. (51), it is clear that the parameter m in this model is related to the power-law index n :

$$m = n-1 \quad (53)$$

In the next section, we will present the solution to the problem for two different cases; the first case is when the viscosity is not a function of temperature, and the second case is when the viscosity-temperature relationship is given by the Reynolds model.

Numerical Solutions

Constant viscosity ($M=0$) : $\mu(\theta) = \mu_o$

In case of constant viscosity Eqn. (44) becomes independent of the temperature and it can be re-written as

$$\frac{du^*}{d\xi} = \beta^{\frac{1}{1+m}} (1-\xi)^{\frac{1}{1+m}} (\sin\alpha)^{\frac{1}{1+m}} \quad (54)$$

Equation (54) subject to condition (48) describes the velocity distribution in ξ -direction under the effects of β , and power m . The solution is possible only when $m \neq -1$, and $m \neq -2$. Thus, using

the boundary condition (48), Eqn. (54) is integrated and we obtain the following solution for the dimensionless velocity

$$u^* = \beta^{\frac{1}{m+1}} \left(\frac{m+1}{m+2} \right) [1 - (1 - \xi)^{\frac{m+2}{m+1}}] (\sin \alpha)^{\frac{1}{1+m}}; \quad m > -1 \quad (55)$$

Now, when $m = 0$ (i.e., when we have a second-grade fluid with constant shear viscosity), Eqn. (55) becomes:

$$u^* = \frac{\beta}{2} (2\xi - \xi^2) \sin \alpha \quad (56)$$

or

$$u = \frac{\rho g h^2 \sin \alpha}{2\mu_o} (2\xi - \xi^2) \quad (57)$$

This expression was given by Gupta (1967). Figure 1 shows the dependence of the dimensionless velocity profiles using power m as a parameter and keeping β constant. The results are given for $m = -0.5, 0$, and 0.5 when $\beta = 4.0$ and $\alpha = 90^\circ$. It is clear that, as m increases the curves shift from right to left. That is, for the case of shear thinning, where m is negative, the velocity looks similar to that of plug flow. For the case of shear thickening, where m is positive, however, the flow seems to be more retarded near the wall. We can also see that the maximum velocity, which is achieved at the free surface, has a higher value for $m = -0.5$ (shear-thinning fluid) than for $m = 0.5$ (shear-thickening fluid).

Effects of parameter β and the inclination angle α are shown in Figs 2 and 3, respectively, for two values of the exponent m . As Figures 2a and 2b show, for a given α , as β increases, the velocity profiles change from an almost uniform (flat) profile to a fully-developed-type profile. For smaller β 's, the profiles are more flat. It is interesting to note that for the same β (for example $\beta = 4.0$) and the same α , the maximum velocity for the case $m = -0.25$ (Fig. 2a) is greater than the maximum velocity for the case $m = 0.25$ (Fig. 2b). That is, for a shear-thinning fluid, the flow achieves a higher velocity at the free surface. For a given β , similar effects are observed when α increases. Again, for the case of shear-thinning fluid, $m = -0.25$, and $\alpha = 90^\circ$ (for example), the maximum velocity is almost 1.7 times the maximum velocity for a shear-thickening fluid with $m = 0.25$.

Figures 4 and 5 show the results for the gradient of the dimensionless velocity at the wall, $(du^*/d\xi)_{\xi=0}$, as a function of β and α , respectively, for $m = -0.25, 0$, and 0.5 . For a given m , $(du^*/d\xi)_{\xi=0}$ increases as β or α increases. This is more drastic when β is less than 1. For a given β , effects of the exponent m on $(du^*/d\xi)_{\xi=0}$ is shown in Fig. 6. When β is less than 1, $(du^*/d\xi)_{\xi=0}$ initially increases as m increases and then it flattens out. When β is greater than one, $(du^*/d\xi)_{\xi=0}$ initially decreases as m increases and gradually it becomes a constant. The dimensionless temperature profiles for the constant viscosity case are also given in Fig. 7. As we can see from Fig. 7a, for a shear-thinning fluid, $m = -0.25$, and a given β , the temperature does not vary much as we change Γ . That is, if we consider Γ as a measure of viscous dissipation, we can see that its effect is more observable for a shear-thickening fluid (see Fig. 7b) than for a shear-thinning fluid. In either case, the temperature profile is almost linear.

Variable Viscosity:

When viscosity depends on temperature, Eqns. (44) and (45) must be solved simultaneously. Typical results for the velocity profiles are shown in Figs 8 to 10 for both shear- thinning and shear-thickening fluids with three values of M . As shown in Fig. 8 we can see that as M increases, the velocity seems to remain the same near the wall and then, at a distance of approximately $\xi = 0.2$ for $m = -0.25$ and $\xi = 0.25$ for $m = 0.25$, the velocity profiles begin to change rapidly, reaching their maximum values at the free stream. We can see that at the same distance from the wall (for example $\xi = 0.8$), and for the same values of $\beta = 1$ and $\Gamma = 1$, the velocity is higher for larger values of M . This is true whether the fluid is shear-thinning or shear-thickening. When β and Γ are both equal to 1, the magnitude of the velocity near the free stream is seen to be almost the same for both shear- thinning and shear-thickening fluids as we can see from Fig. 8. However, this is not the case when either β or Γ is not equal to one. For example, Figure 9 reveals that with $\beta = 0.5$, for shear-thickening fluids, the dimensionless velocity u^* at the free surface is found to be 0.32, 0.44, and 0.52 for M being equal to 0, 1.0, and 1.5, respectively, while for shear-thinning fluids the value of the velocity at the free surface is 0.17, 0.26, and 0.35 for the same values of M . Figure 10 shows a similar trend when $\Gamma = 2$. Comparing Figures 8 and 10 we can see that as Γ changes from 1 to 2, the free surface velocity for a shear-thinning fluid ($m = -0.25$) and for a given $M = 1.5$ is much higher for the larger value of Γ , whereas for a shear-thickening fluid ($m = 0.25$) and for the same $M = 1.5$, the free stream velocity almost remains the same regardless of the value of Γ .

From the energy equation (Eqn. 45), it is clear that when either β or Γ is equal to zero the temperature variation is linear throughout the flow domain. However, when either β or Γ is not equal to zero or when the viscosity is temperature dependent, the temperature distribution becomes highly non-linear as shown in Fig. 11. In this case the calculations reveal the development of a thermal boundary layer near the wall. For $\Gamma = 1.0$, the temperature profiles seem to be almost linear for $m < 0$ or $m > 0$, for a given β and M . This effect is also seen in case of constant viscosity. We can see that as Γ increases, i.e., as the effect of viscous dissipation increases, the temperature rises within the fluid. That is, for a given height, for example $\xi = 0.4$, when $\Gamma = 1.0$, $\theta^* \approx 0.4$, while when $\Gamma = 10.0$, $\theta^* \approx 0.6$ for a shear-thickening fluid ($m = 0.25$), Fig 11b. A similar trend is also observed for the shear-thinning case (Fig. 11a).

Conclusions

The fully developed film flow of a non-Newtonian fluid, modeled as a modified second grade fluid with a temperature dependent viscosity, down an inclined plane is studied. The results in terms of dimensionless velocity and temperature profiles are presented for various dimensionless numbers. The significant parameters in this problem are the angle of inclination α , a dimensionless number β which is related to the Reynolds and Froude numbers, a dimensionless number Γ , a measure of viscous dissipation, which is related to the Eckert and Prandtl numbers, an exponent m which is related to the power-law index n (for a power-law fluid model), and an exponent M which represents the variation of viscosity with temperature. Due to the kinematics of the flow, the normal stress coefficients, α_1 and α_2 , are absorbed in the pressure term, and for this particular problem, they do not affect the velocity and temperature fields directly. The case where viscosity is not a function of temperature, i.e., $M=0$, and where the viscosity is also not shear dependent, i.e., $m = 0$, provides an exact solution for the velocity profile given by Gupta (1967) who also considered the stability of such flow. This exact solution provides a test case for our numerical scheme. We feel that the model proposed here is a general and an important model for non-Newtonian fluids, since it includes the effects of normal stress differences, shear dependent viscosity, and temperature dependent viscosity.

Acknowledgment: We are grateful to Dr. B. Straughan for his helpful and encouraging comments.

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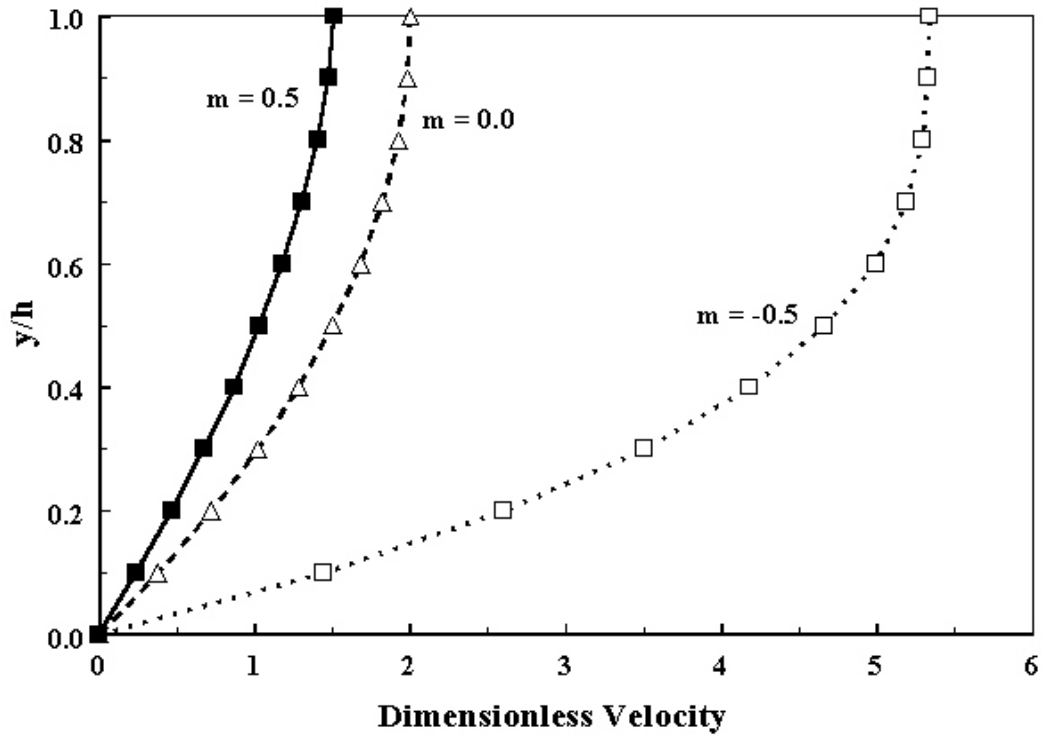


Figure 1. Typical dimensionless velocity profiles calculated for $\beta = 4.0$ and $\alpha = 90^\circ$ with three values of m , (-0.5, 0, and 0.5). The dimensionless velocity u^* was defined by Eqn. (36) (constant viscosity, $M=0.0$).

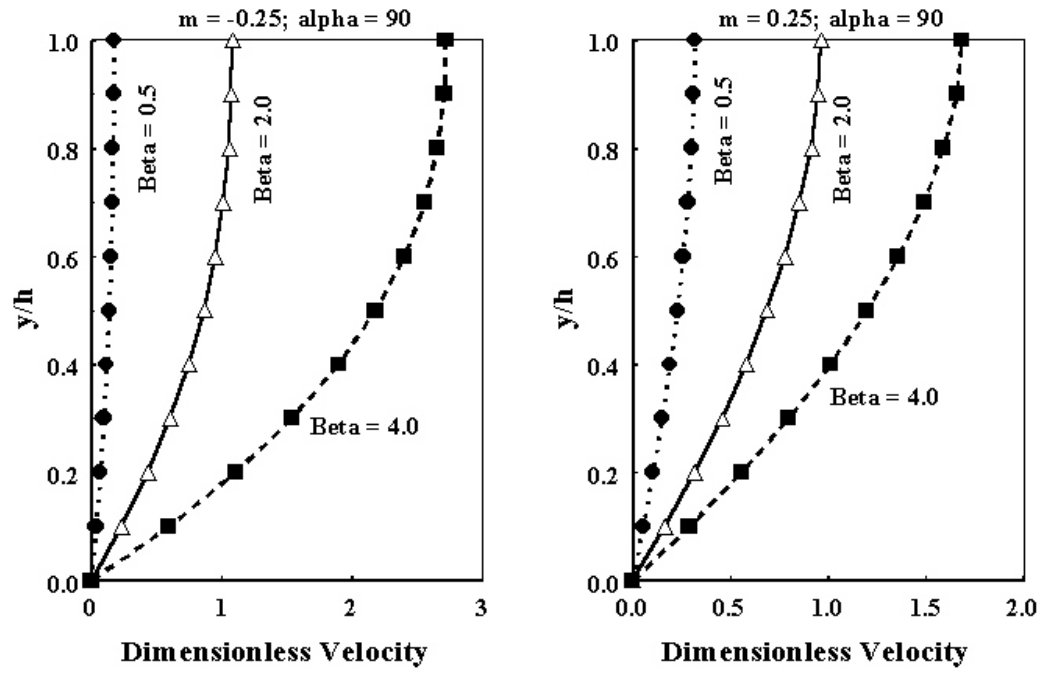


Figure 2. Effect of parameter β on the dimensionless velocity profiles

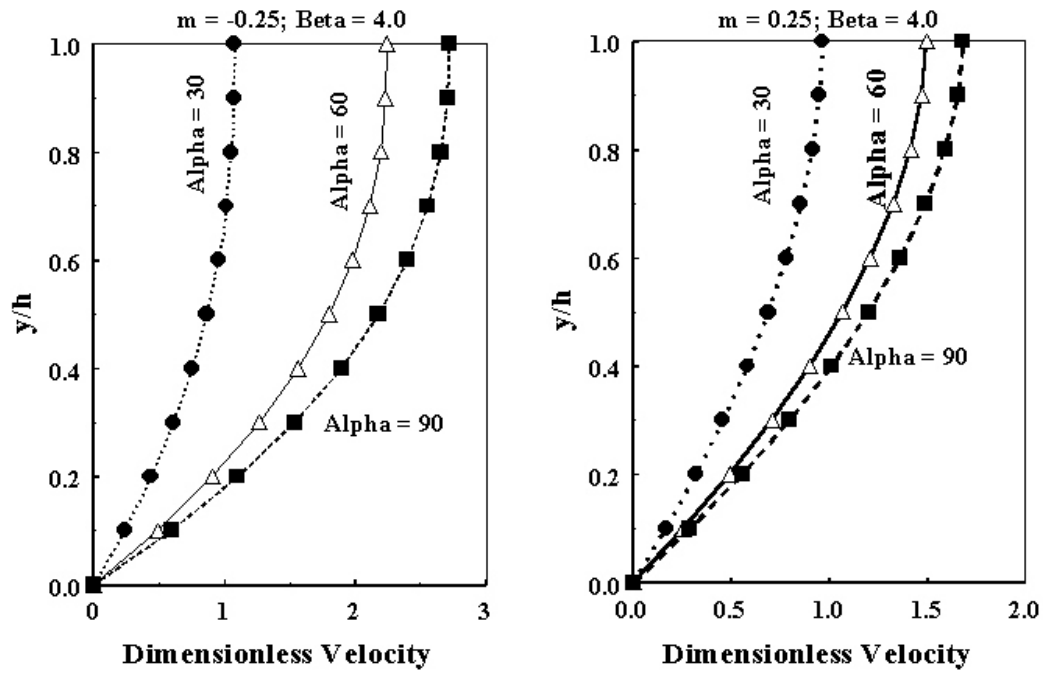


Figure 3. Effect of parameter inclination angle α on the dimensionless velocity profiles

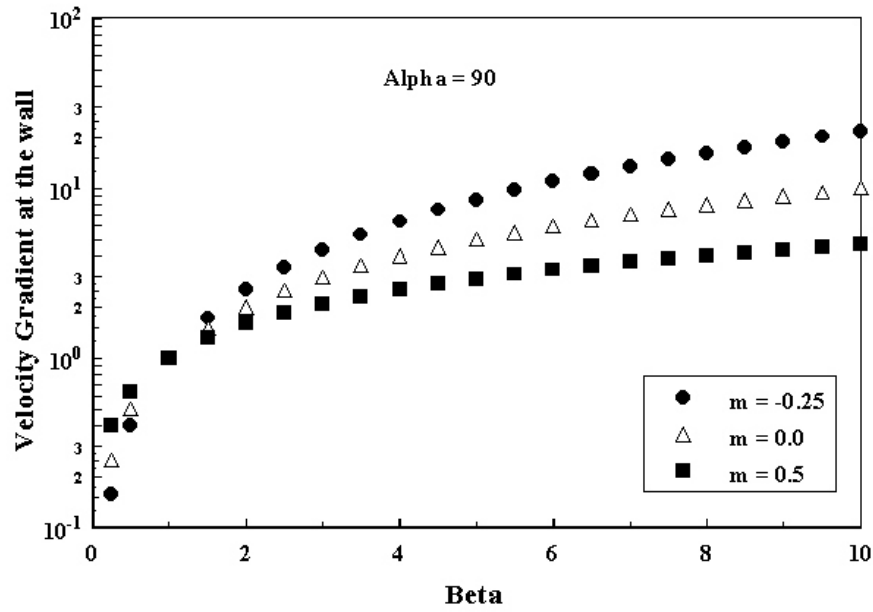


Figure 4. Dimensionless velocity gradient at the surface as a function of parameter β : Effect of the power m ; (constant viscosity case, $M=0$)

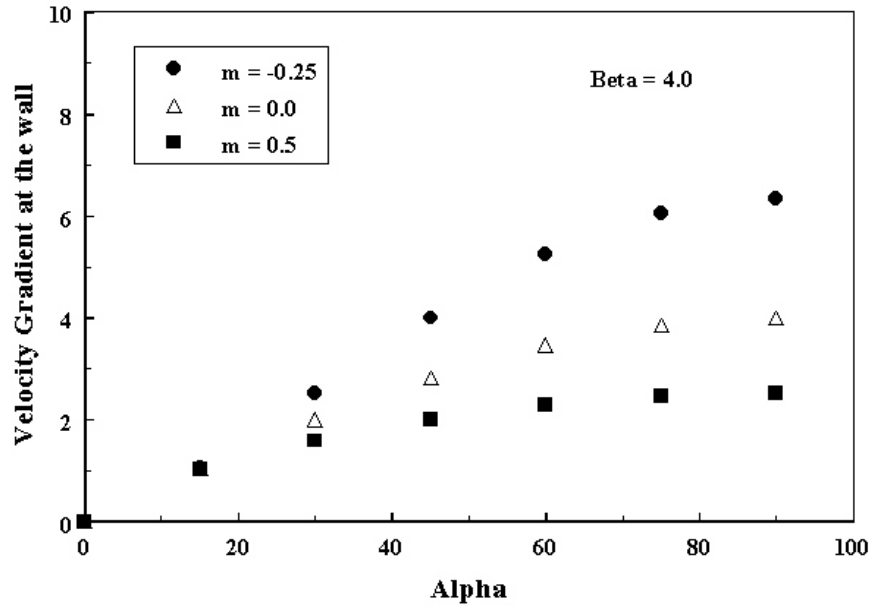


Figure 5. Dimensionless velocity gradient at the surface as a function of inclination α : Effect of the power m ; (constant viscosity case, $M=0$).

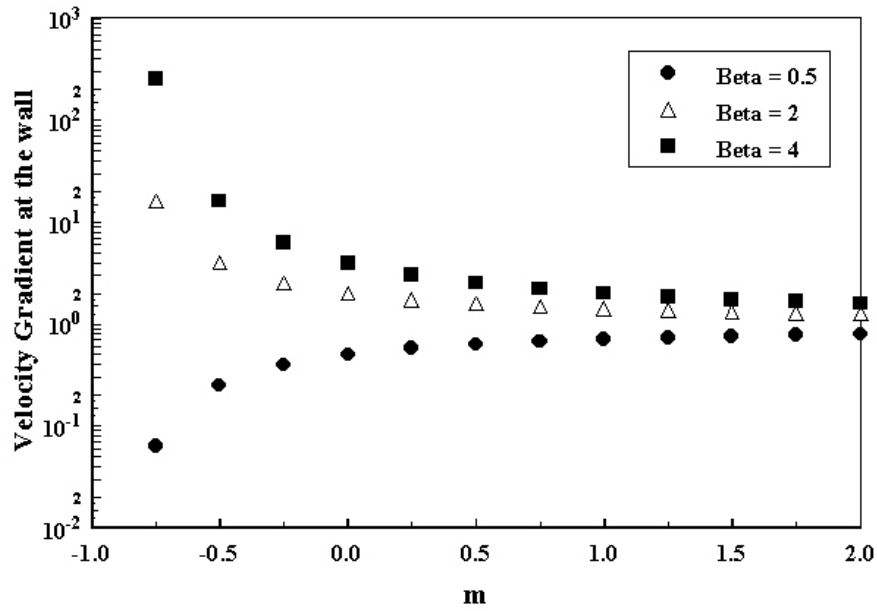


Figure 6 Dimensionless velocity gradient at the surface of the plate as a function of power m : Effect of the β ; (constant viscosity case, $M=0.0$; $\alpha = 90^\circ$).

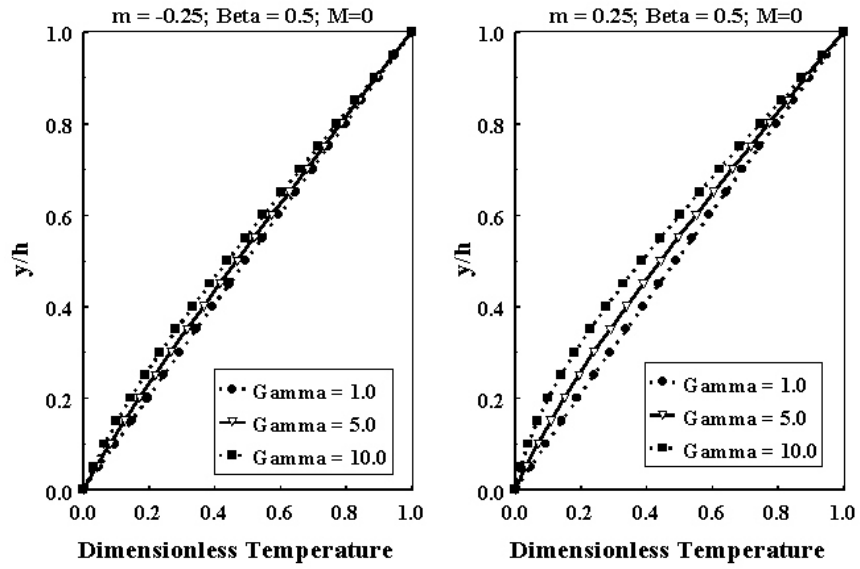


Figure 7. Dimensionless temperature profiles (constant viscosity; $\alpha = 90^\circ$)

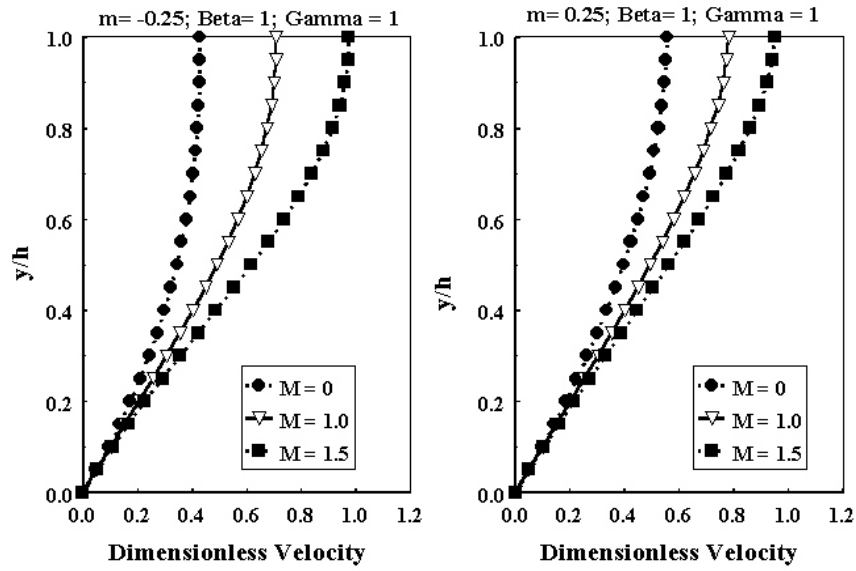


Figure 8. Effects of temperature-dependent viscosity on the dimensionless velocity profile; ($\alpha = 90^\circ$)

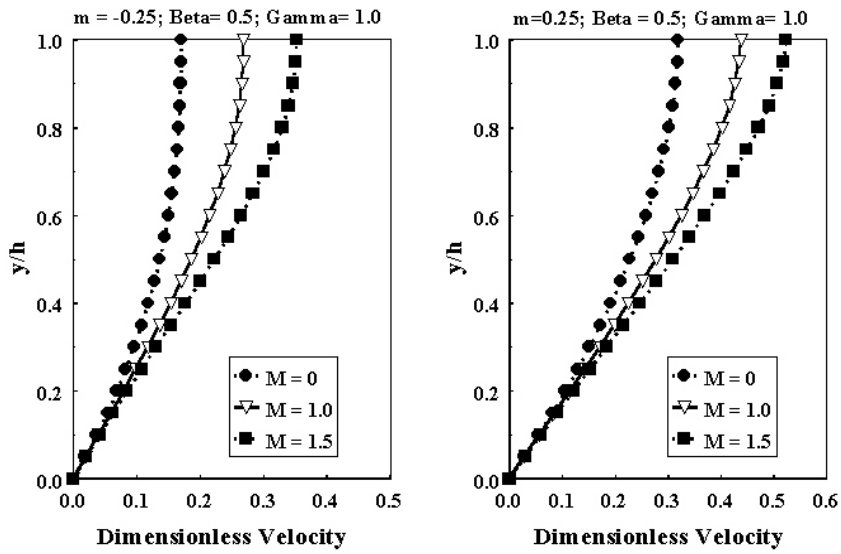


Figure 9. Effects of temperature-dependent viscosity on the dimensionless velocity profile; ($\alpha = 90^\circ$)

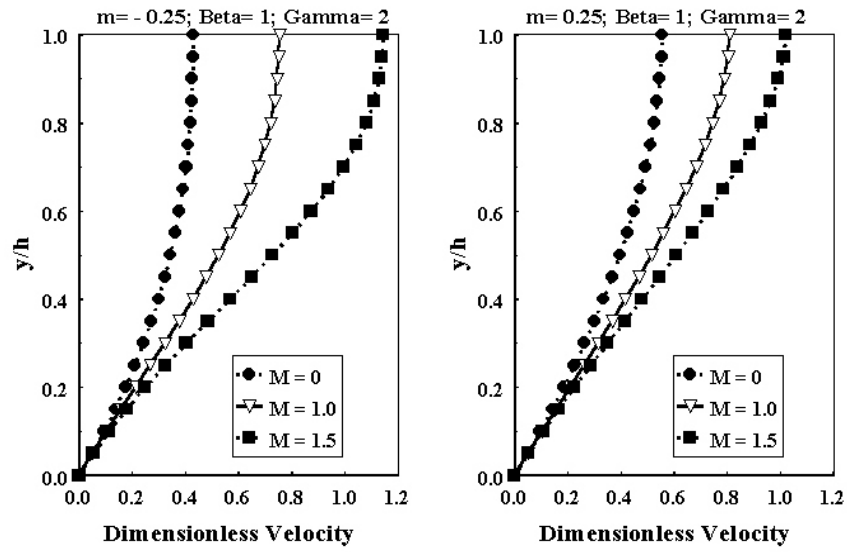


Figure 10. Effects of temperature-dependent viscosity on the dimensionless velocity profile; ($\alpha = 90^\circ$)

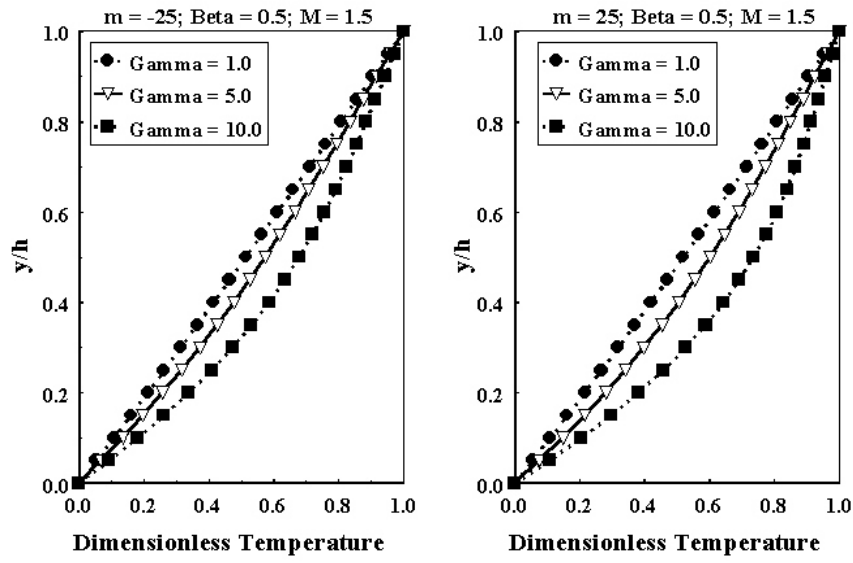


Figure 11. Dimensionless temperature profiles